PHYSICS 100C Final Exam, Thursday, June 11, 8AM-11AM

1. Birefringent crystals have anisotropic optical properties: index of refraction depends on polarization direction. Monochromatic, planar electromagnetic wave with frequency ω is propagating along z-axis of such crystal. Refractive index is n_o for polarization along x ("ordinary ray") and n_e for "extraordinary" polarization direction along y-axis. At position z=0 electromagnetic is linearly polarized at 45 degrees with respect to x and y axis, with electric field amplitude E_0.

- **a)** Write the expression for E (z, t) and B (z, t) fields as a function of time t and z
- **b)** Write the expression for Poynting vector S as a function of time t and z. Calculate time-averaged value $\langle S\rangle_t$.
- **c)** Find the values z_c for which the EM wave is circularly polarized (thickness values of so-called "quarter wave plates").

2. **a)** Derive the formula for critical angle (below which the light is "internally" reflected) for x-rays incident on a material from vacuum, with wavelength λ=0.1 nm. You may assume $N=10^{23}$ free electrons per cm^3 (note non-SI units!), since frequency for x-rays is much greater than all resonance frequencies ($\omega >> \omega_j$). Plug in the numbers and obtain an estimate for the critical angle.

b) Show that the reflectivity of s-polarized x-rays incident on flat vacuum-material interface (from vacuum side) decays as $(q_c/2q)^4$ for large values of q, where $\mathbf{q}=\mathbf{k}_{inc}-\mathbf{k}_{refl}$ is the wavevector transfer defined as the vector difference between incident and reflected wavevectors, while q_c is the value of q corresponding to critical angle.

<u>Hints:</u> Reflection coefficient for s-polarized wave is $R=(1-\alpha\beta)^2/(1+\alpha\beta)^2$, as was derived by you in problem 9.16 - α, β are the same as defined on p. 390 (Eq. 9.106 and 9.110) in the textbook.

You may want to re-define the incident angle with respect to interface plane, rather than interface normal $\psi=\pi/2-\theta$, so that you can conveniently assume in your calculations that critical angle $\psi_c <<1$. You may also want to use variable δ=1-n, and the fact that $\delta <<1$.

3. A monochromatic beam of light with intensity I_0 and frequency ν_0 is incident normally at the perfectly reflective mirror moving away from the beam at (relativistic) velocity v. What are the intensity and frequency of the reflected beam, I_1 and ν_1?

4. **a)** A relativistic electron beam with electron energy 7 GeV is passing through an "undulator" – a device featuring many periodically placed permanent magnets which reverse the direction of magnetic field (applied normal to the direction of the beam) by 180 degrees every d=2.5 cm. Find the wavelength(s) for which the (forward-scattered) radiation resulting from accelerations at each magnetic pole adds up coherently. What part of the electromagnetic spectrum does it belong to?

b) Show that the relative radiative energy losses of an electron moving at velocity v in a circular trajectory with radius R (losses per revolution period, per rest energy mc^2) are proportional, up to a numerical coefficient, to $\beta^3\gamma^4 r_0/R$, where $r_0=e^2/4\pi\varepsilon_0 mc^2$, the classical radius of electron. Identify the missing coefficient.

5. Two thin parallel infinite rods separated by distance *a* carry linear charge density λ and are both moving in the same direction with velocity v along the rod axis. Calculate the force (per unit length) with which the two rods interact in the rest frame and also in the frame of reference moving along with the rods. Compare the two results.

PHYS 100C Final Exam Solutions

Wednesday, June 03, 2009
4:04 PM

***1 A)** $\frac{\omega}{k} = \frac{c}{n} \Rightarrow k = \frac{n\omega}{c}$

$k_o = \frac{n_o \cdot \omega}{c}$, $k_e = \frac{n_e \cdot \omega}{c}$ $\leftarrow$ n is anisotropic therefore different k's for $\hat{x}$ & $\hat{y}$!

Ordinary: $E_x = \frac{E_0}{\sqrt{2}} \cdot e^{i(k_o z - \omega t)}$

$B_y = \frac{E_0 \cdot n_o}{\sqrt{2} \cdot c} e^{i(k_o z - \omega t)}$

Extraordinary: $E_y = \frac{E_0}{\sqrt{2}} \cdot e^{i(k_e z - \omega t)}$

$B_x = -\frac{E_0 \cdot n_e}{\sqrt{2} \cdot c} \cdot e^{i(k_e z - \omega t)}$

where $e^{i(kz - \omega t)}$ stands for $\cos(kz - \omega t)$
(real part)

b) $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$ ($\mu = \mu_0$, changes in n_o / n_e are due to $\varepsilon_o / \varepsilon_e$)

along $\hat{z}$:

$$S_z = \frac{1}{\mu_0}(E_x \cdot B_y - E_y \cdot B_x) = \frac{E_0^2}{2\mu_0 c}\left[n_o e^{i\cdot(2k_o z - 2\omega t)} + n_e \cdot e^{i(2k_e z - 2\omega t)}\right]$$

OR, in cosine notations:

$$S_z = \frac{E_0^2}{2\mu_0 c}\left[n_o \cdot \cos^2(k_o z - \omega t) + n_e \cdot \cos^2(k_e z - \omega t)\right]$$

$$\langle S_z \rangle_t = \frac{E_0^2}{2\mu_0 c} \cdot \frac{n_o + n_e}{2} = \frac{E_0^2 (n_o + n_e)}{4\mu_0 c}$$

OR, since $\mu_0 c = \frac{1}{\varepsilon_0 c}$, $\langle S_z \rangle_t = \frac{\varepsilon_0 c}{4} \cdot E_0^2 (n_o + n_e)$

c) Difference in phases between x & y:

$$\Delta\varphi = (k_o - k_e) \cdot z = \frac{(n_o - n_e)\omega z}{c}$$

Circular polarization: $\Delta\varphi = \frac{\pi}{2} + n\pi$ $n = 0, 1, 2 \ldots$

$$z_c = \frac{\pi c}{2(n_o - n_e)\omega} \cdot (2n + 1)$$

2 A)

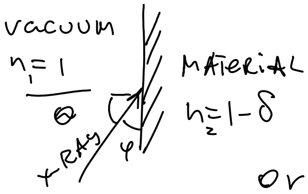
$$n = 1 - \underbrace{\frac{N^* q^2}{2m\varepsilon_0} \sum \frac{f_i}{\omega^2 - \omega_j^2}}_{\delta} = 1 - \delta < 1$$

In Griffith's notations N^* is $\frac{\text{mol.}}{\text{volume}}$, $\sum f_i$ are $\frac{\text{electrons}}{\text{mol.}}$

We will use $N = N^* \cdot \sum f_i$ ← total electrons/volume

Since $\omega \gg \omega_j$ $\quad \delta = \dfrac{N q^2}{2m\varepsilon_0 \omega^2} \approx \dfrac{10^{23} \cdot 10^6 \cdot (10^{-19})^2 (10^{-10})^2}{2 \cdot 10^{-30} \cdot 8 \cdot 10^{-12} (3 \cdot 10^8 \cdot 2\pi)^2} =$

$\approx 5 \cdot 10^{-7}$



Critical Angle:

$$\sin \theta_c = n = 1 - \delta$$

or, if $\varphi = \frac{\pi}{2} - \theta$

$$\cos \varphi_c = 1 - \delta$$

$\varphi_c \ll 1 \quad \cos\varphi_c = 1 - \frac{\varphi_c^2}{2} = 1 - \delta$

$$\varphi_c = \sqrt{2\delta} \quad \Rightarrow \quad \theta_c = \frac{\pi}{2} - \sqrt{2\delta}$$

OR $\quad \varphi_c = \dfrac{q}{\omega}\sqrt{\dfrac{N}{m\varepsilon_0}} \qquad \theta_c = \dfrac{\pi}{2} - \dfrac{q}{\omega}\sqrt{\dfrac{N}{m\varepsilon_0}}$

$\varphi_c \sim \sqrt{10^{-6}} \sim 10^{-3}$ or $1\,\text{mrad} \ll 1$ or 0.06 deg

b)

$$R = \left(\frac{1 - \alpha\beta}{1 + \alpha\beta}\right)^2 \qquad \beta = \frac{n_2}{n_1} = 1 - \delta \quad (\delta \ll 1)$$

$n_1 = 1 \,\|\, n_2 = 1 - \delta \qquad \alpha = \dfrac{\sqrt{1 - \left(\frac{n_1}{n_2} \cdot \sin\theta\right)^2}}{\cos\theta}$

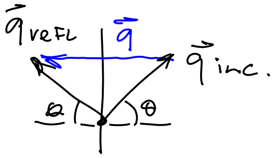
$$\alpha^2 = \frac{1 - \left(\frac{\sin\theta}{1-\delta}\right)^2}{\cos^2\theta} = \frac{1 - (1+\delta)^2 \cdot \sin^2\theta}{\cos^2\theta} \simeq \frac{\overbrace{1 - \sin^2\theta}^{\cos^2\theta} - 2\delta \cdot \sin^2\theta}{\cos^2\theta}$$

$$\alpha^2 = 1 - 2\delta \cdot \tan^2\theta \quad \Rightarrow \quad \alpha \simeq 1 - \delta \cdot \tan^2\theta$$

$$\alpha\beta = (1-\delta)(1 - \delta\tan^2\theta) \simeq 1 - \delta(1 + \tan^2\theta) = 1 - \frac{\delta}{\cos^2\theta}$$

$$1-\alpha\beta = \frac{\delta}{\cos^2\theta} \qquad 1+\alpha\beta = 2+\frac{\delta}{\cos^2\theta} \simeq 2$$

$$R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2 = \frac{\delta^2}{4\cos^4\theta}$$



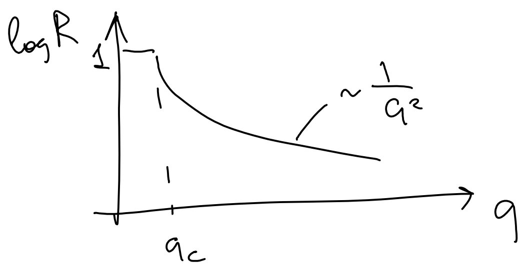
$$|q_{inc}| = |q_{refl}| = \frac{2\pi}{\lambda}$$

$$q = \frac{2\pi}{\lambda}\cdot 2\cos\theta = \frac{4\pi}{\lambda}\cdot\cos\theta$$

If we introduce $q_c = \frac{4\pi}{\lambda}\cdot\cos\theta_c = \frac{4\pi}{\lambda}\cdot\sin\varphi_c = \frac{4\pi}{\lambda}\sqrt{2\delta}$

(or $q_c = \underbrace{\frac{4\pi}{\lambda}\cdot\frac{\overset{\text{el. charge!}}{q_e}}{\omega}}_{\frac{\lambda\cdot\omega}{2\pi}=c}\sqrt{\frac{N}{m\varepsilon_0}} = \frac{2q_e}{c}\sqrt{\frac{N}{m\varepsilon_0}}$ - independent of λ/ω!)

Then: $R = \frac{\delta^2}{4\cos^4\theta} = \left(\frac{q_c}{2q}\right)^4$



3 For observer moving along with mirror the frequency of light

is $\nu' = \gamma\nu_0(1-v/c) = \gamma\nu_0(1-\beta)$

Reflection won't change ν', but in lab frame

$$\nu'' = \gamma\nu'(1-\beta) = \nu_0\cdot\frac{(1-\beta)^2}{1-\beta^2} = \nu_0\cdot\frac{1-\beta}{1+\beta}$$

Intensity:

<u>Method #1</u>: each photon of frequency ν will become ν'' ($h\nu \to h\nu''$)

Time between photons will increase

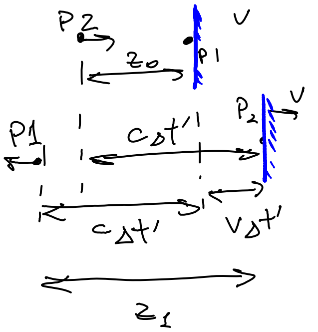
$$\Delta t'' = \Delta t\cdot\frac{1+\beta}{1-\beta}$$

(why: if distance between photons $z = c\cdot\Delta t$

the second photon travels extra distance to "catch" mirror $z_0 + v\Delta t' = c\Delta t'$

$$\Delta t' = \frac{z_0}{c-v}$$

New distance between photons is:



$$z_1 = c\cdot\Delta t' + v\Delta t' = z_0\frac{c+v}{c-v} = z_0\frac{1+\beta}{1-\beta}$$

$$\Delta t'' = \frac{z_1}{c} = \frac{z_0}{c}\cdot\frac{1+\beta}{1-\beta} = \Delta t\,\frac{1+\beta}{1-\beta}$$

Therefore energy/photon is reduced by $\frac{\nu''}{\nu} = \frac{1-\beta}{1+\beta}$

and time between photons increased by

$$\frac{\Delta t''}{\Delta t} = \frac{1+\beta}{1-\beta}$$

total intensity decrease:

$$\frac{I_1}{I_0} = \frac{h\nu''}{h\nu}\cdot\left(\frac{\Delta t''}{\Delta t}\right)^{-1} \left(= \frac{1-\beta}{1+\beta}\right)^2$$

Method #2

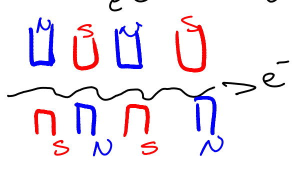
Lorentz transform For E, B: $E' = \gamma(E + \beta\cdot B) = \sqrt{\frac{1-\beta}{1+\beta}}\cdot E$

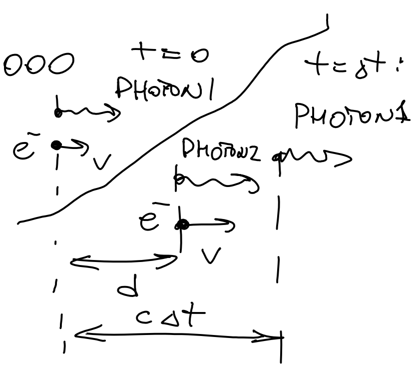
$$E'' = \sqrt{\frac{1-\beta}{1+\beta}}\cdot E' = \frac{1-\beta}{1+\beta}E$$

Since $I \sim E^2$

$$\frac{I_1}{I} = \left(\frac{E''}{E}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2$$

4 A) $\gamma = \frac{E}{m_e\cdot c^2} = \frac{7\,GeV}{0.5\,MeV} = 14{,}000$





$d = v\cdot\Delta t$

distance between two photons:

$$c\cdot\Delta t - d = d\left(\frac{c}{v} - 1\right) \approx d(1-\beta)$$

photons in-phase when $d(1-\beta) = n\lambda$

$$1-\beta \simeq \frac{(1-\beta)(1+\beta)}{2} = \frac{1-\beta^2}{2} = \frac{1}{2\gamma^2}$$

$$\frac{d}{2\gamma^2} = n\lambda \qquad n = 1, 2, 3, \ldots$$

$$\boxed{\lambda = \frac{d}{2\gamma^2 n}}$$

FOR $n=1$

$$\lambda_1 = \frac{d}{2\gamma^2} = 0.63\,\text{Å} \quad (\text{X-RAY})$$

b) From Lienard's formula:

$$P = \frac{\mu_0 e^2 \gamma^6}{6\pi c}\cdot\left(a^2 - \left|\frac{\bar{v}\times\bar{a}}{c}\right|^2\right)$$

$$a = \frac{v^2}{R}, \quad \bar{v}\times\bar{a} = \frac{v^3}{R} \Rightarrow a^2 - \left(\frac{\bar{v}\times\bar{a}}{c}\right)^2 = \frac{v^4}{R^2} - \frac{v^6}{R^2c^2}$$

$$P = \frac{\mu_0 \cdot e^2\gamma^6}{6\pi c}\cdot\frac{v^4}{R^2}\underbrace{\left(1-\frac{v^2}{c^2}\right)}_{1/\gamma^2} = \frac{\mu_0 e^2}{6\pi R^2}\gamma^4\cdot\beta^4\cdot c^3$$

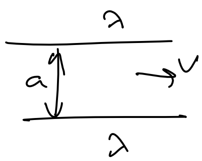
$$\underset{\text{per period}}{\Delta E} = P\cdot\frac{2\pi R}{v} = \frac{\mu_0 e^2 c^2}{3\cdot R}\cdot\gamma^4\beta^3$$

Norm. to rest mass:

$$\frac{\Delta E}{mc^2} = \frac{\overbrace{\mu_0 c^2}^{1/\varepsilon_0}}{3}\cdot\frac{(e^2/mc^2)}{R}\cdot\gamma^4\beta^3 =$$

$$= \frac{4\pi}{3}\cdot\frac{r_0}{R}\cdot\gamma^4\beta^3$$

5.



Repulsive force.

In moving frame:

$$E = \frac{\lambda}{2\pi\varepsilon_0\cdot a} \quad (\text{Gauss' law for cylinder})$$

$$\frac{\partial F}{\partial l} = \frac{\lambda^2}{2\pi\varepsilon_0\cdot a} = \frac{\gamma^2\lambda^2}{2\pi\varepsilon_0\cdot a}$$

In rest frame, el. force: $\frac{\partial F_e}{\partial l} = \frac{\lambda'^2}{2\pi\varepsilon_0\cdot a}$

Magn. field: $B = \frac{\mu_0\cdot v\cdot\lambda'}{2\pi a}$ (Ampere's law)

$$\frac{\partial F_M}{\partial l} = B\cdot v\cdot\lambda' = \frac{\mu_0 v^2\lambda'^2}{2\pi a} = \frac{1}{2\pi\varepsilon_0}\cdot\frac{v^2}{c^2}\cdot\frac{\lambda'^2}{a}$$

$$\frac{\partial F_{TOT}}{\partial \ell'} = \frac{\partial F_e}{\partial \ell'} + \frac{\partial F_m}{\partial \ell'} = \frac{\lambda'^2}{2\pi\varepsilon_0 a}\left(1 - v^2/c^2\right) = \frac{\lambda'^2}{2\pi\varepsilon_0 \cdot a\,\gamma^2}$$

But since $\lambda' = \gamma\lambda$

$$\frac{\partial F_{TOT}}{\partial \ell} = \frac{\lambda^2 \cdot \gamma^2}{2\pi\varepsilon_0 \cdot a \cdot \gamma^2} = \frac{\lambda^2}{2\pi\varepsilon_0 \cdot a} = \frac{\partial F}{\partial \ell'}$$ (in moving FRAME)